Introduction

In the thermal theory of foundry one considers two basic macroscopic solidification models it means the solidification in constant temperature (the Stefan problem) and solidification in an interval of temperature. In the first case the mathematical model reduces to the system of two parabolic equations and along the moving boundary the well known Stefan condition is given. In the second case three changing with time sub-domains should be taken into account. The latent heat is evolved in the interval of temperature and adequate sub-domain is called mushy zone. The cast iron solidification proceeds in more complex way and the process partially takes place in constant temperature and partially in an interval of temperature. The enthalpy – temperature diagram illustrating this course of solidification for cast iron (3.9% C) is shown in Figure 2. The computations concerning the analysis of cast iron solidification can be realized only on the basis of numerical
methods and very effective in this case is the generalization of so-called Alternating Phase Truncation Method. As it was mentioned the searched solution was obtained on the basis of BEM.

**Governing Equations**

Let us consider the heterogenous multi-phase material $D = D_1(t) \cup D_2(t) \cup D_3(t)$ with moving boundaries $\xi_1(t), \xi_2(t)$. Sub-domains $D_m(t), m = 1, 2, 3$ correspond to solid state, mushy zone and liquid part of casting whereas $\xi_2(t), \xi_1(t)$ are the moving boundaries between liquid state and mushy zone and between mushy zone and solid state. It should be pointed that $\xi_1(t)$ corresponds to eutectic temperature $T_e$. The latent heat of this transition is equal to $L_e [J/kg]$ – Figure 1.

![Fig. 1. Casting sub-domains](image)

The mathematical description of cast iron cooling and solidification can be assumed in the following from:

- **Energy equations for sub-domains** $D_m(t)$

\[
x \in D_m(t) : c_m(T) \rho_m(T) \frac{\partial T(x,t)}{\partial t} = \text{div} \left[ \lambda_m(T) \nabla T(x,t) \right], \quad m = 1, 2, 3
\]

where $c_m(T), \rho_m(T), \lambda_m(T)$ are the specific heat, mass density and thermal conductivity of $D_m(t)$ sub-domain, $T, x, t$ are the temperature, spatial coordinates and time.

- **Boundary condition for** $x \in \xi_1(t)$:

\[
x \in \xi_1(t) : \begin{cases}
-\lambda_2 \nabla \cdot \nabla T_2(x,t) = -\lambda_1 \nabla \cdot \nabla T_1(x,t) + L_e \rho_e \nu_{ne} \\
T_2(x,t) = T_1(x,t) = T_e
\end{cases}
\]
where $\bar{n} \cdot \text{grad} T$ is a normal derivative at the point $x \in \xi_1(t)$, $T_e$ – eutectic temperature, $v_{ne}$ – rate of solidification in normal direction.

- Boundary condition for $x \in \xi_2(t)$

$$
x \in \xi_2(t) : \begin{cases}
    -\lambda_3 \bar{n} \cdot \text{grad} T_3(x, t) = -\lambda_2 \bar{n} \cdot \text{grad} T_2(x, t) \\
    T_3(x, t) = T_2(x, t) = T_L
\end{cases}
$$

where $T_L$ is the liquidus temperature.

The condition (2) is called the Stefan boundary condition whereas (3) it is a typical 4th kind boundary condition for the case of ideal contact between considered sub-domains.

- Boundary condition on the outer surface of the system

$$
x \in \Gamma_0 : \Phi[T(x, t), \bar{n} \cdot \text{grad} T(x, t)] = 0
$$

In this model the 1st kind (temperature) and 2nd kind (heat flux) boundary conditions have been taken into account.

- Initial condition

$$
t = 0 : \quad T_m(x, 0) = T_{m0}(x), \quad m = 1, 2, 3
$$

In order to introduce the GAPT algorithm the mathematical model must be rebuilt to the so-called enthalpy convention.

The physical enthalpy of alloy related to an unit of volume is defined as follows

$$
H(T) = \int_0^T c(\mu) \rho(\mu) d\mu + \eta(T) L_e \rho_e
$$

where

$$
c(T) \rho(T) = \begin{cases}
    c_1 \rho_1, & T < T_e, \\
    c_2 \rho_2, & T \in [T_e, T_L], \\
    c_3 \rho_3, & T > T_L,
\end{cases}
\eta(T) = \begin{cases}
    0, & T < T_e \\
    1, & T \geq T_e
\end{cases}
$$

So, the new mathematical model is of the form:

$$
x \in D_m(t) : \frac{\partial H(x, t)}{\partial t} = \text{div}[a_m \text{grad} H(x, t)], \quad m = 1, 2, 3
$$
Fig. 2. Enthalpy – temperature function for 3.9% C

Fig 3. The approximation of enthalpy temperature function
Boundary Element Method for 2D Parabolic Equation

Let us consider a certain homogeneous domain $D$ with thermal diffusivity $a$, it means

$$x \in D : \frac{\partial H(x,t)}{\partial t} = a \text{ div } [\nabla H(x,t)]$$

(9)

In the case of an object oriented in rectangular co-ordinate system the last equation can be expressed as follows

$$x \in D : \frac{\partial H(x,t)}{\partial t} = a \nabla^2 H(x,t) = a \left( \frac{\partial^2 H}{\partial x_1^2} + \frac{\partial^2 H}{\partial x_2^2} \right)$$

(10)

The integral equation corresponding to the linear parabolic equation (10) is of the form [5]

$$C(\xi)H(\xi, t^F) = \int_{t^0}^{t^F} \left[ q^*(\xi, x, t^F, t) H(x, t) - H^*(\xi, x, t^F, t) q(x, t) \right] d\Gamma dt +$$

$$+ \int_{D} H^*(\xi, x, t^F, t^0) H(x, t^0) dD$$

(11)

where $[t^0, t^F]$ it the considered interval of time, $C(\xi)$ is a certain coefficient associated with the position of considered point $x$, it means $C(\xi) = 1$ for internal point, $C(\xi) = 0.5$ for a smooth boundary point $\xi$ etc., $H^*(\xi, x, t^F, t)$is a fundamental solution. For 2D problems and the object oriented in rectangular co-ordinate system this function is of the form [6]
\[ H^*(\xi, x, t^F, t) = \frac{1}{4\pi a(t^F - t)} \exp \left[ \frac{-r^2}{4a(t^F - t)} \right] \]  

where \( r \) is the distance from considered point \( x = \{x_1, x_2\} \) to the point \( \xi = \{\xi_1, \xi_2\} \) where concentrated heat source is applied, at the same time the adequate formula determining the heat flux can be found

\[ q^*(\xi, x, t^F, t) = -a \frac{\partial H^*}{\partial n} = \frac{d}{8\pi a(t^F - t)^2} \exp \left[ \frac{-r^2}{4a(t^F - t)} \right] \]  

where

\[ d = (x_1 - \xi_1)n_1 + (x_2 - \xi_2)n_2 \]

and \( n_1, n_2 \) - direction cosines of the boundary normal vector in the considered point \( (x_1, x_2) \). In this way discussed previously mathematical model solidification and cooling process can be replaced by the integral equation (11) in which the given initial and boundary conditions should be taken into account.

In order to obtain a solution of equation (11) a time mesh is defined

\[ 0 = t^0 < t^1 < \ldots < t^{f-1} < t^f < \ldots < t^F < \infty \]  

and \( \Delta t = t^f - t^{f-1} = \text{const.} \) Using the 1st scheme of BEM the enthalpy field for the moment \( t^f \) is obtained only on the basis of known enthalpy values at the moment \( t^{f-1} \).

The boundary \( \Gamma \) is divided into \( N \) boundary elements \( \Gamma_j, j = 1, 2, \ldots N \) and the interior \( D \) is divided into \( L \) internal cells \( D_l, l = 1, 2, \ldots L \). Then a numerical approximation of equation (11) is of the form

\[ C(\xi^i)H(\xi^j, t^f) = \sum_{j=1}^{N} \int_{\Gamma_j} [ \int_{t^{f-1}}^{t^f} q^*(\xi^i, x^j, t^f, t^{f-1}) dt ] H(x^j, t^f) d\Gamma_j + \]

\[ \sum_{j=1}^{N} \int_{\Gamma_j} [ \int_{t^{f-1}}^{t^f} H^*(\xi^i, x^j, t^f, t^{f-1}) dt ] q(x^j, t^f) d\Gamma_j + \]

\[ \sum_{l=1}^{L} \int_{D_l} H^*(\xi^i, x^j, t^f, t^{f-1}) H(x^j, t^{f-1}) dD_l \]
Time integration is possible in an analytic way whereas the integrals with respect to $\Gamma_j$ and $D_l$ must be calculated numerically. The final system of algebraic equations can be written in the form

$$\sum_{j=1}^{N} G_{ij} q(x^j, t^j) = \sum_{j=1}^{N} Z_{ij} H(x^j, t^j) + \sum_{l=1}^{L} P_{il} H(x^i, t^{l-1})$$  \hspace{1cm} (17)

where

$$G_{ij} = \frac{1}{4\pi a} \int_{\Gamma_j} \frac{d^2}{dr^2} \left( \frac{r_i^2}{4a\Delta t} \right) d\Gamma_j, \quad Z_{ij} = \frac{1}{2\pi} \int_{\Gamma_j} \frac{d_{ij}}{r_i^2} \exp \left( \frac{-r_i^2}{4a\Delta t} \right) d\Gamma_j$$  \hspace{1cm} (18)

and $E_i(-)$ is the exponential integral function. In the numerical solution the function $E_i(-)$ is approximated by interpolating polynomials.

From the system (17) unknown values of "missing" enthalpy or heat fluxes in the boundary nodes are found using Gauss elimination method with pivoting and next the enthalpy field for internal nodes can be calculated from the following formula

$$H(x^i, t^i) = \sum_{j=1}^{N} Z_{ij} H(x^j, t^j) - \sum_{j=1}^{N} G_{ij} q(x^j, t^j) + \sum_{l=1}^{L} P_{il} H(x^i, t^{l-1})$$  \hspace{1cm} (19)

The determined enthalpy distribution constitutes a pseudo-initial condition for calculations concerning the next time step.

**Alternating Phase Truncation Method**

Let us assume that $t^{l+1}$ and $t^l$ denote two distinguished levels of time. The value $\Delta t = t^l - t^{l-1}$ constitutes so-called time mesh step and as a rule $\Delta t = \text{const}$. It should be pointed in this place that GAPTM can co-operate with optional numerical algorithm of energy equation approximate solution in which discretization of time is introduced.
Additionally it is assumed that enthalpy distribution in area $D$ at the moment $t_{f-1}$ is known and the function $H = H(T)$ is of the form presented in Figure 3.

The first stage of computations concerns the hottest phase (molten metal). The enthalpy distribution in domain $D$ at the moment $t_{f-1}$ is transformed in this way

$$V_3(x, t_{f-1}) = \max (A_4, H(x, t_{f-1}))$$ (20)

This new pseudo–initial condition corresponds to conventional reduction of whole area $D$ to the molten metal domain. Next on the basis of numerical methods (e.g. BEM) the transition $t_{f-1} \rightarrow t'_f$ is calculated. The solution for a moment $t'_f : V_3^*(x, t'_f)$ is corrected according to the formula

$$V_3(x, t'_f) = V_3^*(x, t_{f-1}) + H(x, t_{f-1}) - V_3(x, t_{f-1})$$ (21)

and the first stage of computations is finished.

In the second step the whole area $D$ is reduced to the mushy zone. A reduction of area $D$ to domain $D_2$ corresponds to acceptance of the following pseudo–initial condition

$$V_2(x, t_{f-1}) = \min \{A_3, \max (A_2, V_3(x, t'_f))\}$$ (22)

It is easy to check that $V_2(x, t_{f-1}) \in [A_2, A_3]$.

If $V_2^*(x, t'_f)$ denotes the enthalpy field calculated in this stage then the final effect of stage realization arises from formula

$$V_2(x, t'_f) = V_2^*(x, t'_f) + V_3(x, t'_f) - V_2(x, t_{f-1})$$ (23)

At the last stage (solid state) the following pseudo–initial condition is assumed

$$V_1(x, t_{f-1}) = \min [A_1, V_2(x, t'_f)]$$ (24)

After the computation of $V_1^*(x, t'_f)$ one obtains

$$H(x, t'_f) = V_1^*(x, t'_f) + V_2(x, t'_f) - (x, t_{f-1})$$ (25)

The solution $H(x, t'_f)$ found in this stage constitutes the searched enthalpy field at the moment $t = t'_f$. 

The transition $t^{i-1} \rightarrow t^i$ requires the solving three linear diffusion problems in conventionally homogeneous domains but in this way the well known difficulties with numerical simulation of typical moving boundary problems can be "passed round".

**Example of Numerical Simulation**

The casting (plate) with dimensions 0,01x0,05 [m] treated as 2D object made from cast iron (3,9% C) has been considered. The enthalpy distribution it means the course of function $H = H(T)$ has been constructed on the basis of enthalpy – carbon diagram [7]. So, the course presented in Figure 2 has been taken into account. The mathematical description of successive fragments of considered function was assumed in the form [9]

$$H = \begin{cases} 
4932.6T & T < 1145 \\
5648010, 6570000, & T = 1145 \\
6570000 + 12734.85(T - 1145), & T \in (1145, 1300) \\
8543900 + 5475(T - 1300), & T \geq 1300 
\end{cases}$$

It should be pointed that values of enthalpy resulting from Figure 2 were multiplied by adequate mass densities for successive sub-domains of considered material and that the dimension of above equation is [kJ/m$^3$].

The symmetrical fragment of plate has been considered (Fig. 4).

![Fig. 4. The mesh](image)

The problem was solved adiabatic boundary conditions for $x_1 = 0$ and $x_2 = 0$ and the 1st kind condition (Schwarz solution [8]) for the rest of the boundary.

The results of temperature field in domain $D$ are presented in Figure 5 and numerical simulation (cooling curves for distinguished points 1 and 2 – Fig. 4) in Figure 6.
The next stage of investigations will concern the real boundary conditions and mold influence analysis and this problem requires a construction of BEM algorithm for heterogeneous area with essentially different thermophysical parameters.

Fig. 5. Temperature distribution after 0.2 s, 1s, 3s and 5 s
Fig. 6. Cooling curves at the points 1 and 2

References

Streszczenie

MODEL MAKROSKOPOWY KRZEPNIĘCIA ŻELIWA
ROZWIĄZANIE NUMERYCZNE ZA POMOCĄ MEB

Przedstawiono model numeryczny krzepnięcia żeliwa (zadanie 2D), wykorzystujący połączenie I schematu metody elementów brzegowych dla równań parabolicznych z uogólnioną metodą przemiennej fazy. Zawarto opis matematyczny procesu krzepnięcia żeliwa, algorytm obliczeń oraz rezultaty symulacji numerycznej do odlewu płyty (zadanie 2D).