PRESSURE DROPS IN CONICAL FLOW OF MOLTEN METAL

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ABSTRACT
Practical moulding processes involve geometrically complex dies. Such dies are usually tapered or streamlined, e.g., conical dies, to achieve maximum output rate under conditions of laminar flow. These conical or converging flows may be analysed in terms of their extensional and simple shear components to calculate the relationships between volume flow rate and pressure drop.

The model of viscoplastic fluid is used and obtained results have been illustrated by examples of conical flows the Ostwald-de Waele and the Bingham fluids.

1. INTRODUCTION

Conical or more general converging flows occur frequently in many industrial applications, particularly manufacturing processes involving metal melts and in those instances the prediction of pressure losses is often of paramount concern.

The manufacturing is often realized in die-casting machines where the molten metal form a mixture of liquid metal with solidified particles [7]. The flow of this mixture may be modeled as the flow of viscoplastic (non-Newtonian) fluids [2, 5, 6].

To describe the rheological behaviour of these fluids the Ostwald-de Waele and Bingham models are often used; the constitutive equations are the following:
- for the model of Ostwald-de Waele (power-law fluid):

\[ \tau = (\mu \dot{\gamma})^m \]  
(1.1)

- for the model of Bingham:

\[ \tau = \tau_0 + (\mu \dot{\gamma}) \]  
(1.2)

Here \( \dot{\gamma} \) is the shear stress, \( \tau_0 \) is the yield shear stress, \( \mu \) is the coefficient of plastic viscosity, \( m \) is nonlinearity index, \( \dot{\gamma} \) is the shear rate.

Note that if \( m = 1 \) the Ostwald-de Waele fluid reduces to the classical Newtonian fluid.
By their very nature, converging flows necessarily involve a component of extensional flow and the behaviour of fluids in extensional flows may therefore be of vital importance to the successful simulation of the flows under consideration. Cogswell [1,3] was perhaps the first to recognize of extensional viscosity in converging flows. A developed version of Cogswell's analysis was used by Walicki and Walicka [4] in a study of conical converging flow. More general analysis of this problem is suggested by the same authors in the work [8].

There are three basic modes of deformation: bulk deformation, simple shear deformation and simple tension. For the purposes of this analysis we assume that the bulk deformations are sufficiently small to be neglected. Simple shear deformation takes place when the streamlines are parallel, but correspond to different material velocities. Extensional flow occurs when the streamlines through a cross-section correspond to constant material velocities, but are not parallel.

Basing on the considerations suggested in the work [8] the authors present in this paper an approximate analysis for converging flow of a molten metal with axial symmetry, where the flow is coni-cylindrical as at a change in cross section in pipe flow (Fig 1).

2. FLOW IN CONICAL DIES

2.1. Pressure drop due to telescopic shear within the die

Consider an axi-symmetric flow through conical die (Fig.1). Spherical coordinates \((R, \theta, \phi)\) will be used, \(z\) being the axis of symmetry and principal direction of flow. Corresponding velocity component is \(u_z\) which is parallel to the axis of \(z\).

![Fig. 1 Schematic diagram of conical flow.](image1)

![Fig. 2 Schematic diagram of forces for the analysis of the pressure drop due to telescopic shear.](image2)

Consider an element of length \(dz\) (Fig.2) laying within the die. For this element we may resolve the forces parallel to the center line of the die and obtain

\[
\pi R^2 dp_z = 2\pi R \frac{dz}{\cos \theta_0} \tau_w \cos \theta_0
\]

hence

\[
dp_z = \frac{2\tau_w}{R} dz \quad \text{where} \quad dz = dR \ctg \theta_0
\]
Then

\[ dp_s = 2 \cot g \theta_0 \frac{\tau_w}{R} dR \]  \hspace{1cm} (2.1)

Here \( p_s \) is the static pressure, \( \tau_w \) is the local wall shear stress.

Integrating Eq (2.1) across the die in the interval \( (R_i, R_o) \) we have

\[ \Delta p_s = 2 \cot g \theta_0 \int_{R_i}^{R_o} \frac{\tau_w}{R} dR \]  \hspace{1cm} (2.2)

The local wall shear stress is equal [8]:

\[ \tau_w = \frac{R}{2} \frac{dp}{dz} = \tau_0 \]  \hspace{1cm} (2.3)

Finally one has

\[ \Delta p_s = -2 \cot g \theta_0 \int_{R_i}^{R_o} \frac{dp}{dz} dR \] \hspace{1cm} or \hspace{1cm} \[ \Delta p_s = 2 \tau_0 \cot g \theta_0 \int_{R_i}^{R_o} Y dR \]  \hspace{1cm} (2.4)

Here \( Y \) is the function connected with the flow rate \( Q \).

2.2. Pressure drop due to extensional flow within the die.

Let a small element of length \( dz \) be acted on by an average extensional stress \( \sigma_c \) parallel to the axis of symmetry of the die (Fig 3).

Denoting by \( p_e \) the pressure due to extension one obtains

\[ \pi R^2 dp_e = \left[ \pi (R + dR)^2 - \pi R^2 \right] \sigma_c \]
and hence

$$dp_e = 2\sigma_e \frac{dR}{R}$$  \hspace{1cm} (2.5)

Treating the phenomenon of extensional flow purely in form, we can write for local extensional stress [8]:

$$\sigma = L \dot{\varepsilon}$$  \hspace{1cm} (2.6)

where $L$ - is the function of viscosity and $\dot{\varepsilon}$ is the local strain rate in the $R$-direction. To define the local strain rate $\dot{\varepsilon}$ consider a converging annulus within the die of thickness $h$ and radius $r$ (hatched on Fig 4). Let the angle of convergence be $\theta$. For uniform convergence one has

$$\dot{\varepsilon} = \frac{1}{2\pi rh} \frac{d}{dt} (2\pi rh) = \frac{2}{\pi} \frac{dr}{t}$$

but

$$\frac{2}{\pi} \frac{dr}{t} = \frac{du_z}{dr} = \frac{du_z}{dr} = \frac{du_z}{dr} = \frac{du_z}{dr} = \frac{dr}{R} \frac{dR}{dr}$$

where $u_z$ is the velocity parallel to the center line of the medium in the annulus.

Finally one has

$$\dot{\varepsilon} = -\frac{du_z}{dr} \frac{R}{\pi} \frac{dR}{dr}$$

Then the local extensional stress is

$$\sigma = L \frac{du_z}{dr} \frac{R}{\pi} \frac{dR}{dr}$$  \hspace{1cm} (2.9)

The average extensional stress through the cross section of radius $R$ is

$$\sigma_e = \frac{1}{2\pi R^2} \int_0^R 2\pi \sigma rdr$$

Having $L$ and $\dot{\varepsilon}$ one may calculate first $\sigma$ and next the average stress $\sigma_e$ from the expression (2.10) and then pressure drop due to extensional flow from the equation:

$$\Delta p_e = 2 \int_{R_1}^{R_2} \sigma_e \frac{dR}{R}$$  \hspace{1cm} (2.11)
3. EXAMPLES OF APPLICATION

For power-law fluid one has [4]:

\[ Y_p = \frac{1}{2} \frac{d \rho}{dz} \left[ \frac{\mu(3m+1)Q}{m\pi R^3} \right]^m R^{-1}, \]

\[ L_p = \mu (r Y_p)^{1 - \frac{1}{m}}, \]

\[ \sigma = \left[ \frac{\mu(3m+1)Q}{m\pi R^3} \right]^m \left( \frac{r}{R} \right)^2 \tan \theta \]

and the pressure drops are as follows:

\[ \Delta p = \frac{2 \mu g \theta_0}{3m} \left[ \frac{\mu(3m+1)Q}{m\pi R^3} \right]^m \left( 1 - \beta^{3m} \right), \quad (3.1) \]

\[ \Delta p_c = \frac{2 \mu g \theta_0}{3m} \left[ \frac{\mu(3m+1)Q}{m\pi R^3} \right]^m \left( 1 - \beta^{3m} \right), \quad (3.2) \]

where \( \beta \) is the inverse of the contraction ratio, i.e., \( \beta = R_i / R_0 \). The total pressure drop in the die for power-law fluid is

\[ \Delta p = \Delta p_c + \Delta p_e, \quad (3.3) \]

For the Bingham fluid one has [8]:

\[ L = \mu \frac{Y}{Y - 1}, \quad \sigma = \frac{\tau_0 Y \tan \theta_0}{R^2} \left( \frac{r}{R} \right)^2, \quad \sigma_e = \frac{\tau_0 Y \tan \theta_0}{2}. \]

The value of \( Y \) is given by the solution of the following equation:

\[ 3Y^4 - 4(3K + 1)Y^3 + 1 = 0 \]

where

\[ K = \frac{\mu Q}{\pi R^3 \tau_0}. \]

For large value of \( K \) (or small value of \( \tau_0 \)) one may assume
\[ Y = \frac{4\mu Q}{\pi R^3 \tau_0} + \frac{4}{3} \]

and the pressure drops are as follows:

\[ \Delta p_s = \frac{8 \cotg \theta_0}{3} \left[ \frac{\mu Q}{\pi R^3} \left( 1 - \beta^3 \right) - \tau_0 \ln \beta \right] \quad (3.4) \]

\[ \Delta p_c = \frac{4 \cotg \theta_0}{3} \left[ \frac{\mu Q}{\pi R^3} \left( 1 - \beta^3 \right) - \tau_0 \ln \beta \right] \quad (3.5) \]

For small value of \( K \) (or large value of \( \tau_0 \)) one has

\[ Y = 1 + \left( \frac{2\mu Q}{\pi R^3 \tau_0} \right)^{\frac{1}{2}} \]

and the pressure drops are as follows:

\[ \Delta p_s = \frac{4 \cotg \theta_0}{3} \left[ \left( \frac{2\mu Q \tau_0}{\pi R^3} \right)^{\frac{1}{2}} \left( 1 - \beta^2 \right) - \frac{3}{2} \frac{\tau_0}{\ln \beta} \right] \quad (3.6) \]

\[ \Delta p_c = \frac{2 \cotg \theta_0}{3} \left[ \left( \frac{2\mu Q \tau_0}{\pi R^3} \right)^{\frac{1}{2}} \left( 1 - \beta^2 \right) - \frac{3}{2} \frac{\tau_0}{\ln \beta} \right] \quad (3.7) \]

Note that here the value of \( K \) is connected with the width of quasi-solid core flow in which the molten metal moves with a constant velocity; if \( K \) increases this width decreases. The total pressure drop in this case also is given by the expression (3.3).

The formulae (3.3) for the total pressure drop for both the fluids may be re-written as:

\[ \Delta p = \zeta \frac{\rho v_i^2}{2} \quad (3.8) \]

where \( \zeta \) - is the coefficient of local pressure drop,
\( \rho \) - the fluid density,
\( v_i \) - the velocity in the \( R_i \) cross section of the die; this velocity is given by

\[ v_i = \frac{Q}{\pi R_i^2} \quad (3.9) \]

then coefficient \( \zeta \) is equal to
for power-law fluid, and

\[ \zeta = \frac{16}{3 \text{Re}} \left( 2 \cot \theta_0 + \tan \theta_0 \right) \left( 1 - \beta^3 - \frac{1}{2} \frac{\text{B} \log \beta}{\text{B} \log \beta} \right) \tag{3.11} \]

\[ \zeta = \frac{4}{3 \text{Re}} \left( 2 \cot \theta_0 + \tan \theta_0 \right) \left( 1 - \beta^3 - \frac{3}{4} \left( \frac{\text{B} \log \beta}{\text{B} \log \beta} \right) \right) \tag{3.12} \]

for the two cases of Bingham fluid, respectively.

Here

\[ \text{Re} = \frac{\rho u^2 - m (2 \text{R}_1)^m}{\mu^m} \quad \text{or} \quad \text{Re} = \frac{\rho u^2 (2 \text{R}_1)}{\mu} \quad \text{and} \quad \text{B} \log \beta = \frac{\tau_0 (2 \text{R}_1)}{\mu \text{u}_1} \tag{3.13} \]

are the Reynolds and Bingham numbers.

In the plastic and molten metal processing is frequently accepted the following formula [5]:

\[ \Delta \varphi = \frac{8 \mu Q \cot \theta_0 \left( 1 - \beta^3 \right)}{3 \pi \text{R}_1} \tag{3.14} \]

as the first approximation for the pressure drop in the die. It is clear that this formula represents only the pressure drop due to the shear flow of Newtonian fluid.

4. CONCLUSION

An approximate method for the solution of a molten metal flow in converging cones has been put forward.

The analysis and hence the results of the preceding sections are subject to several assumptions and approximations, the most severe of which is perhaps the neglect of the radial velocity component but these simplifications do not appear to alter the qualitative validity of the study.

A simple trigonometric approach allows the calculation of pressure drop within the conical die. The principal of the analysis is sufficiently simple that it can be extended to other, more complex geometries while still obtaining a tractable solution.

REFERENCES


SPADKI CIŚNENIA W STOŻKOWYM PRZEPŁYWIE STOPIONEGO METALU

ABSTRAKT

Techniczne procesy odlewu wymagają geometrycznie złożonych zwężek. Takie zwężki są zazwyczaj zbieżne lub prostoliniowe, jak np. stożkowe zwężki, aby osiągnąć maksimum wydatku dla warunków przepływu laminarnego. Te stożkowe lub zbieżne przepływy można analizować w zależności od rozciągających i ścinających składowych, co pozwala wyliczyć zależności między wydatkiem objętościowym a stratą ciśnienia.

Użyto modelu lepkoplastycznego płynu, a wyniki zilustrowano przykładami przepływów płynu Ostwala - de Waele i płynu Binghama.