SENSITIVITY ANALYSIS OF SOLIDIFICATION WITH RESPECT TO THE MOULD THICKNESS

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SUMMARY

In the paper the sensitivity analysis of solidification process with respect to the mould thickness is presented. On the stage of numerical computations the boundary element method is used. In the final part of the paper the results obtained are shown.

Key words: solidification, numerical modelling, shape sensitivity analysis

1. MATERIAL DERIVATIVE

Among different sensitivity problems, especially important are the shape sensitivity ones [1]. They consist in finding the sensitivity of structural response to variations in the initial shape of the body. Let us consider the 1D casting-mould system. We assumed that b=L is the shape design parameter – Figure 1. Using the concept of material derivative we can write [1, 2]

\[
\frac{\Delta T}{\Delta b} = \frac{\partial T}{\partial b} + \frac{\partial T}{\partial x} v
\]

(1)

where T(x, t) is the temperature, v=v(x, b) is the velocity associated with design parameter b.

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Because (c.f. equation (1))

\[
\frac{D}{D^T} \left( \frac{\partial T}{\partial x} \right) = \frac{\partial}{\partial b} \left( \frac{\partial T}{\partial x} \right) + \frac{\partial^2 T}{\partial x^2} = \frac{\partial}{\partial b} \left( \frac{\partial T}{\partial x} \right) + \frac{\partial^2 T}{\partial x^2}
\]

(2)

and

\[
\frac{\partial}{\partial x} \left( \frac{DT}{Db} \right) = \frac{\partial}{\partial x} \left( \frac{\partial T}{\partial x} \right) + \frac{\partial^2 T}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{DT}{Db} \right) + \frac{\partial^2 T}{\partial x^2} \frac{\partial T}{\partial x} + \frac{\partial T}{\partial x} \frac{\partial^2 T}{\partial x^2} \frac{\partial}{\partial x}
\]

(3)

therefore

\[
\frac{D}{D^b} \left( \frac{\partial T}{\partial x} \right) = \frac{\partial}{\partial t} \left( \frac{DT}{Db} \right) - \frac{\partial T}{\partial x} \frac{\partial}{\partial x} \frac{\partial T}{\partial x}
\]

(4)

In a similar way one obtains

\[
\frac{D}{D^b} \left( \frac{\partial T}{\partial t} \right) = \frac{\partial}{\partial t} \left( \frac{DT}{Db} \right)
\]

(5)

Using formula (4) we have

\[
\frac{D}{D^b} \left( \frac{\partial^2 T}{\partial x^2} \right) = \frac{D}{D^b} \left[ \frac{\partial}{\partial x} \left( \frac{\partial T}{\partial x} \right) \right] = \frac{\partial}{\partial x} \left[ \frac{D}{D^b} \left( \frac{\partial T}{\partial x} \right) \right] - \frac{\partial^2 T}{\partial x^2} \frac{\partial}{\partial x}
\]

(6)

and next

\[
\frac{D}{D^b} \left( \frac{\partial^2 T}{\partial x^2} \right) = \frac{\partial}{\partial x} \left[ \frac{D}{D^b} \left( \frac{\partial T}{\partial x} \right) \right] - \frac{\partial^2 T}{\partial x^2} \frac{\partial}{\partial x} = \frac{\partial^2}{\partial x^2} \left( \frac{DT}{Db} \right) - 2 \frac{\partial^2 T}{\partial x^2} \frac{\partial T}{\partial x} - \frac{\partial^2 T}{\partial x^2} \frac{\partial}{\partial x}
\]

(7)

Presented above formulas are necessary in order to realize the shape sensitivity analysis of solidification process.

2. SOLIDIFICATION MODEL

The 1D casting-mould system is considered - Figure 1. The temperature field in casting sub-domain determines the energy equation.
where \( C(T) \) is the substitute thermal capacity [3], \( \lambda \) is the thermal conductivity. A temperature field in mould sub-domain describes the equation of the form

\[
0 < x < L_i : \quad C(T) \frac{\partial T(x,t)}{\partial t} = \lambda \frac{\partial^2 T(x,t)}{\partial x^2} \tag{8}
\]

and

\[
L_i < x < L : \quad c_m \frac{\partial T_m(x,t)}{\partial t} = \lambda_m \frac{\partial^2 T_m(x,t)}{\partial x^2} \tag{9}
\]

On the contact surface between casting and mould the continuity condition

\[
x = L_i : \begin{cases}
-\lambda \frac{\partial T(x,t)}{\partial x} = \lambda_m \frac{\partial T_m(x,t)}{\partial x} \\
T(x,t) = T_m(x,t)
\end{cases} \tag{10}
\]

is assumed. The remaining boundary conditions are following

\[
\left\{ \begin{array}{l}
x = 0 : \quad \lambda \frac{\partial T(x,t)}{\partial x} = 0 \\
x = L : \quad -\lambda_m \frac{\partial T_m(x,t)}{\partial x} = 0
\end{array} \right. \tag{11}
\]

For the moment \( t = 0 \):

\[
T(x,0) = T_0 \quad T_m(x,0) = T_{mo} \tag{12}
\]

3. SHAPE SENSITIVITY ANALYSIS – DIRECT APPROACH

If the direct approach of sensitivity method is applied [1, 2] then the governing equations are differentiated with respect to shape parameter \( b \). So, the differentiation of equation (8) gives (c.f. formulas (5) and (7))

\[
\frac{D C(T)}{Db} \frac{\partial T(x,t)}{\partial t} + C(T) \frac{\partial U(x,t)}{\partial t} = \\
\lambda \left[ \frac{\partial^2 U(x,t)}{\partial x^2} - 2 \frac{\partial^2 T(x,t)}{\partial x^2} \frac{\partial v}{\partial x} - \frac{\partial T(x,t)}{\partial x} \frac{\partial^2 v}{\partial x^2} \right] \tag{13}
\]

Using the dependence (8) and assuming that the substitute thermal capacity is described by staircase function [3] one obtains
where \( U(x, t) = \frac{D_T}{D_b} \) is the sensitivity function.

In similar way we differentiate the equation (9) and then

\[
\frac{c_m}{\partial t} \frac{\partial U_m(x, t)}{\partial t} = \frac{\partial U(x, t)}{\partial x} = 2 \frac{\partial T(x, t)}{\partial x} - \frac{\partial^2 U(x, t)}{\partial x^2} \tag{15}\]

where \( U_m(x, t) = \frac{DT_m}{Db} \).

Differentiation of equation (10), (11), (12) leads to the following conditions (c.f. formula (4))

\[
x = L : \left\{ -\lambda \left[ \frac{\partial U(x, t)}{\partial x} - \frac{\partial T(x, t)}{\partial x} \right] = \lambda_m \left[ \frac{\partial U_m(x, t)}{\partial x} - \frac{\partial T_m(x, t)}{\partial x} \right] \right. \tag{16}
\]

and

\[
x = 0 : \quad \lambda \left[ \frac{\partial U(x, t)}{\partial x} \right] = 0 \]
\[
x = L : \quad -\lambda_m \left[ \frac{\partial U_m(x, t)}{\partial x} \right] = 0 \tag{17}\]

while

\[
t = 0 : \quad U(x, 0) = 0 \quad U_m(x, t) = 0 \tag{18}\]

In order to realize the shape sensitivity analysis of solidification process with respect to the mould thickness, the following definition of velocity associated with design parameter \( b = L \) can be accepted

\[
v = \begin{cases} 0, & 0 \leq x \leq L_t \\ \frac{x - L_t}{b - L_t}, & L_t \leq x \leq L \end{cases} \tag{19}\]

The equations connected with the sensitivity functions \( U(x, t) \) and \( U_m(x, t) \) have the following form (c.f. equation (14))

\[
0 < x < L_t : \quad C(T) \frac{\partial U(x, t)}{\partial t} = \lambda \frac{\partial^2 U(x, t)}{\partial x^2} \tag{20}\]
and (c.f. equation (15))

\[ L_1 < x < L: \quad c_m \frac{\partial U_m(x,t)}{\partial t} = \lambda_m \frac{\partial^2 U_m(x,t)}{\partial x^2} + \frac{2c_m}{b - L_1} \frac{\partial T_m(x,t)}{\partial t} \]  

(21)

On the contact surface between casting and mould we have

\[
\begin{align*}
\frac{-\lambda_m}{c_m} \frac{\partial U(x,t)}{\partial x} &= \frac{\partial U_m(x,t)}{\partial x} - q_m(x,t) \\
U(x,t) &= U_m(x,t)
\end{align*}
\]

(22)

The remaining conditions (17) and (18) are not changing.

4. RESULTS OF COMPUTATIONS

The basic and additional problems have been solved using the 1st scheme of the boundary element method supplemented by the temporary temperature field correction procedure [3, 4]. The 1D casting-mould system of dimensions \(2L_1=0.02 \text{ m}\) (casting) and \(0.03 \text{ m}\) (mould) has been considered. The following input data have been introduced: \(\lambda=35 \text{ W/mK}\), \(\lambda_m=2.6 \text{ W/mK}\), \(c_S=5.175 \times 10^6 \text{ J/m}^3\text{K}\), \(c_p=1.118 \times 10^8\), \(c_L=5.74 \times 10^6\), \(c_m=1.75 \times 10^6\), pouring temperature \(T_0=1570 \text{ °C}\), liquidus temperature \(T_L=1505 \text{ °C}\), solidus temperature \(T_S=1470 \text{ °C}\), initial mould temperature \(T_m=30 \text{ °C}\). In Figure 2 the sensitivity function multiplied by the change of mould thickness \(\Delta L=0.005 \text{ m}\) for times 5, 10, 15, ..., 50 [s] is shown.

![Fig. 2. Sensitivity function for times 5, 10, ..., 50 [s](Rys. 2. Funkcja wrażliwości dla czasów 5, 10, ..., 50 [s])](image-url)
Figure 3 illustrates the heating curves at the points $x=0.015$ [m] and $x=0.025$ [m] for basic value of the mould thickness and for the lower and upper values of this parameter.

It is visible, than in this case the change of mould thickness causes the maximum change of temperature about 110°C.

REFERENCES


ANALIZA WRAZLIWOŚCI PROCESU KRZEPNIĘCIA NA ZMIANY WYMIARÓW FORMY ODLEWNICZEJ

STRESZCZENIE

W pracy przedstawiono analizę wrażliwości procesu krzepnięcia ze względu na grubość formy odlewniczej Na etapie obliczeń wykorzystano metodę elementów brzegowych. W końcowej części artykułu pokazano wyniki obliczeń.

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