IDENTIFICATION OF ALLOY LATENT HEAT ON THE BASIS OF MOULD TEMPERATURE (PART 1)

B. MOCHNACKI¹, J.S. SUCHY²

1 Institute of Mathematics and Computer Science, Czestochowa University of Technology, Częstochowa, ul. Dąbrowskiego 73, Poland
2 Faculty of Foundry Engineering, AGH Cracow, ul. Reymonta 23, Poland

SUMMARY

In the paper the inverse problem consisting in the identification of volumetric latent heat of alloy analyzed on the basis of additional information concerning the transient temperature field in the mould sub-domain is discussed. From the practical point of view the problem seems to be interesting because the measurements realized in this region of casting-mould system are simpler (low temperatures, knowledge of sensors exact position etc.). The identification is realized under the assumption that the substitute thermal capacity of alloy (this parameter is determined by the value of latent heat) is assumed to be a constant value or (in the second variant) a linear function. In the first part of the paper the theoretical problems are presented.

Key words: inverse problems, modeling of solidification, numerical methods.

1. INTRODUCTION

The energy equation describing the casting solidification is the following (only conductional heat transfer is considered)

\[
c(T) \frac{\partial T(x,t)}{\partial t} = \nabla \left[ \lambda(T) \nabla T(x,t) \right] + L \frac{\partial f_s(x,t)}{\partial t}
\]  

(1)

¹ prof. dr hab. inż., moch@imi.pcz.pl
² prof. dr hab. inż., jsuchy@agh.edu.pl
where \( c(T) \) is a volumetric specific heat, \( \lambda(T) \) is a thermal conductivity, \( L \) is a volumetric latent heat, \( f_s \) is a volumetric solid state fraction at the considered point, \( T, x, t \) denote the temperature, geometrical co-ordinates and time.

The source function in the energy equation (1) can be eliminated (the one domain approach [1, 2]) and finally one obtains the equation

\[
C(T) \frac{\partial T(x,t)}{\partial t} = \nabla \left[ \lambda(T) \nabla T(x,t) \right]
\]

(2)

where \( C(T) = c(T) - L \mathrm{d} f_s / \mathrm{d} T \) is a substitute thermal capacity.

The equation (1) is supplemented by the equation (or equations) concerning a mould sub-domain

\[
c_m (T) \frac{\partial T_m(x,t)}{\partial t} = \nabla \left[ \lambda_m(T) \nabla T_m(x,t) \right]
\]

(3)

where \( c_m \) is a mould volumetric specific heat, \( \lambda_m \) is a mould thermal conductivity.

In the case of typical sand moulds on the contact surface between casting and mould the continuity condition in the form

\[
-\lambda \frac{\partial T(x,t)}{\partial n} = -\lambda_m \frac{\partial T_m(x,t)}{\partial n}
\]

(4)

\[ T(x,t) = T_m(x,t) \]

can be accepted (\( \partial / \partial n \) denotes a normal derivative).

Additionally on the external surface of the system the condition in general form

\[
\Phi \left[ T_m(x,t), \frac{\partial T_m(x,t)}{\partial n} \right] = 0
\]

(5)

is given. For time \( t = 0 \) the initial values

\[ t = 0: \quad T(x,0) = T_o(x), \quad T_m(x,0) = T_{m0}(x) \]

(6)

are also known.

It is self-evident that for molten metal and solidified part of casting \( f_s = 0 \), \( f_s = 1 \) and then \( \mathrm{d} f_s / \mathrm{d} T = 0 \). Summing up, the equation (2) describes the thermal processes in the whole, conventionally homogeneous, casting domain.

For instance, the function fulfilling the above formulated conditions can be assumed in the form

\[
\Phi \left[ T_m(x,t), \frac{\partial T_m(x,t)}{\partial n} \right] = 0
\]
where \( \frac{T_L - T}{T_L - T_S} \) is the mushy zone volumetric specific heat. In the special case (\( p = 1 \)) we have \( C(T) = c_p + \frac{L}{(T_L - T_S)} \) and the substitute thermal capacity of mushy zone is a constant value.

The other possibility of substitute thermal capacity definition results from the ‘direct’ approach. The form of function \( C(T) \) is assumed a’priori, for example [1]

\[
C(T) = c_S + a(T - T_S), \quad T \in [T_S, T_L]
\]

where \( c_S \) is a volumetric specific heat of solid, while a parameter \( a \) can be found on the basis of condition

\[
\int_{T_S}^{T_L} C(T) \, dT = c_p \left(T_L - T_S\right) + L
\]

The presented above hypotheses concerning the function \( C(T) \) will be used for identification of volumetric latent heat \( L \).

2. SENSITIVITY ANALYSIS

The method of inverse problems solution discussed in this paper require the formulation of sensitivity model with respect to volumetric latent heat. The construction of sensitivity model consists in the differentiation of the basic equation and conditions with respect to the parameter analyzed (direct approach [3]). In order to simplify the further consideration, the constant value of casting thermal conductivity is assumed. Such simplification is not necessary and it is possible to find the sensitivity equation for \( \lambda = \lambda(T) \). So
\[ C(T) \frac{\partial U(x, t)}{\partial t} = \lambda \nabla^2 U(x, t) - \frac{\partial C(T)}{\partial L} \frac{\partial T(x, t)}{\partial t} \]  

(12)

where \( U = \partial T / \partial L \). For the first version of substitute thermal capacity approximation (a constant value) we have

\[
C(T) = \begin{cases} 
    c_S & T < T_S \\
    c_p + \frac{L}{T_L - T_S}, & \frac{\partial C(T)}{\partial L} = \frac{1}{T_L - T_S}, & T \in [T_S, T_L] \\
    c_L & T > T_L 
\end{cases}
\]

(13)

In the case of linear approximation of \( C(T) \) (equation (10)) assuming \( c_p \approx c_S \) (in order to simplify the final formula) we have

\[
C(T) = c_S + \frac{2L}{(T_L - T_S)^2} [T(x, t) - T_S] \quad T \in [T_S, T_L]
\]

(14)

and

\[
\frac{\partial C(T)}{\partial L} = \begin{cases} 
    0 & T < T_S \\
    \frac{2}{(T_L - T_S)^2} [T(x, t) - T_S + LU(x, t)] & T \in [T_S, T_L] \\
    0 & T > T_L 
\end{cases}
\]

(15)

For the mould sub-domain (assuming the constant values of \( c_m, \lambda_m \)):

\[
c_m \frac{\partial U_m(x, t)}{\partial t} = \lambda_m \nabla^2 U_m(x, t)
\]

(16)

where \( U_m = \partial T_m / \partial L \). The continuity condition (4) leads to the formula

\[
\left\{ \begin{array}{c}
    -\lambda \frac{\partial U(x, t)}{\partial n} = -\lambda_m \frac{\partial U_m(x, t)}{\partial n} \\
    U(x, t) = U_m(x, t)
\end{array} \right.
\]

(17)

while the condition (5) takes a form

\[
\Phi_s \left[ U_m(x, t), \frac{\partial U_m(x, t)}{\partial n} \right] = 0
\]

(18)

For time \( t = 0 \) the initial condition is given:
3. INVERSE PROBLEMS

If the inverse problem is solved it is necessary to have an additional information concerning the course of the process. So, let us assume that at the selected set of points from casting-mould domain the cooling (heating) curves are known (in other words the values $T_{d_i}^f$ at the points $x_i$ for times $t^f$)

$$T_{d_i}^f = T_d(x_i, t^f), \quad i = 1, 2, \ldots, M, \quad f = 1, 2, \ldots, F$$

(20)

As was mentioned, on the stage of numerical computations (see: part 2 of the paper) the sensors have been located only in the mould sub-domain this means beyond the area in which the evolution of latent heat takes place.

In order to solve the inverse problem, the least squares criterion is applied [4, 5, 6]

$$S = \sum_{i=1}^{M} \sum_{j=1}^{F} (T_i^f - T_{d_i}^f)^2$$

(21)

where $T_i^f = T(x_i, t^f)$ is the calculated temperature at the point $x_i$ for time $t^f$ for arbitrary assumed value of $L$.

Differentiating the criterion (21) with respect to the unknown volumetric latent heat and using the necessary condition of minimum, one obtains

$$\frac{\partial S}{\partial L} = 2 \sum_{i=1}^{M} \sum_{j=1}^{F} (T_i^f - T_{d_i}^f) \frac{\partial T_i^f}{\partial L} \bigg|_{L=L^k} = 0$$

(22)

where $k$ is the number of iteration, $L^k$ for $k=0$ is the arbitrary assumed value of $L$, while $L^k$ for $k>0$ results from the previous iteration.

Function $T_i^f$ is expanded in a Taylor series about known value of $L^k$ this means

$$T_i^f \approx (U^f)^k + \left. \frac{\partial T_i^f}{\partial L} \right|_{L=L^k} (L^{k+1} - L^k) = (U^f)^k + (U^f)^k \left( L^{k+1} - L^k \right)$$

(23)

Putting (23) into (22) one has

$$\sum_{i=1}^{M} \sum_{j=1}^{F} \left[ (U_i^f)^k \right]^2 (L^{k+1} - L^k) = \sum_{i=1}^{M} \sum_{j=1}^{F} (U_i^f)^k \left[ T_i^f - (U_i^f)^k \right]$$

(24)

this means
This equation allows to find the value of $L^{k+1}$. The iteration process is stopped when the assumed accuracy is achieved or after the achieving the assumed value of iterations.

For each iteration the basic problem and additional one connected with the sensitivity function should be solved using the numerical methods (e.g. FDM - see: part 2). The parameter $L$ can be also identified using the simpler numerical algorithm. The criterion (21) is calculated for the successive values of $C(T)$ (this means for the successive values of $L$) with a certain step $\Delta C(T)$ starting from $C(T) = c_L$ (or $c_k$). The value of functional $S$ successively decreases and next begins to increase. At this region we introduce the smaller step $\Delta C(T)$ and the procedure is continued. This simplified method of minimum localization is possible because only one parameter appearing in the inverse problem is unknown and we know the good start point and the direction of changes (see: part 2).
IDENTYFIKACJA UTAJONEGO CIEPŁA KRZEPNIĘCIA NA PODSTAWIE POLA TEMPERATURY W FORMIE ODLEWNICZEJ (CZĘŚĆ 1)

STRESZCZENIE

W pracy przedstawiono rozwiązanie zadania odwrotne polegającego na identyfikacji utajonego ciepła krzepnięcia stopu Al-Si (odniesionego do jednostki objętości) na podstawie krzywych nagrzewania w wybranych punktach z obszaru masy formierskiej. Z praktycznego punktu widzenia problem wydaje się interesujący, ponieważ pomiary temperatury w masie formierskiej są prostsze i dokładniejsze niż pomiary temperatury w obszarze krzepnięcia odlewu. W rozważaniach teoretycznych założono ogólną znajomość przebiegu zastępczej pojemności cieplnej stopu (funkcja stała, funkcja liniowa). W części pierwszej artykułu przedstawiono podstawy teoretyczne rozpatrywanego problemu.

Recenzował Prof. Ryszard Parkitny