Analysis of Spherical Particles Size Distribution – Theoretical Basis

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Abstract

The determination of the form of a probability density function (PDF$_3$) of diameters for nodular particles by using a probability density function (PDF$_3$), which form is empirically estimated from cross-sections of these nodules in a metallographic specimen, can be regarded as a special case of Wicksell’s corpuscle problem (WCP). The estimation of the PDF$_3$ for the nodular particles provides information about the kinetics of these particles nucleation, and so about the kinetics of their growth. This information is essential for building more accurate mathematical models of the alloy crystallization.

In the paper there are presented two derivations of the methods used for the estimation of the PDF$_3$ form. The first method bases on diameters received from a planar cross-section. The second one uses also data from the planar cross-section but not the diameters only chords. Both methods provide practical rules for the analysis of the empirical diameters’ and chord’s size distribution and allow to estimate the mean value of the external surface area of the particles.

Keywords: Estimation of a diameter size distribution, Planimetric analysis, Linear analysis

List of variables used in the text

C1 – subclass of C2 including only these cut nodules with the radius $r_2 \geq t$ which centres lie closer than $\sqrt{r_2^2 - t^2}$ on both sites of the cutting plane
C2 – subclass of C3 including only these cut nodules the distance of which from their centre and the cutting plane on both sites is less than $r_3$
C3 – class of nodules with the radii ranging from $r_3$ to $r_3 + dr_3$

$dn_{C1}$– the cardinality of the set C1
$dn_{C2}$– the cardinality of the set C2
$dn_{C3}$– the cardinality of the set C3

$E[r_2]$– the expected value of nodule radii
$E[S_3]$– the expected value of the outer nodule surface areas

$F_1(t)$– cumulative distribution function (CDF$_1$) of random chords with the length $2 \cdot r_1 \leq 2 \cdot t$ lying on the cutting plane

$F_2(t)$– cumulative distribution function (CDF$_2$) of random intersection radii with the length $r_2 \leq t$ lying on the cutting plane

$F_3(t)$– cumulative distribution function (CDF$_3$) of nodule radii with the length $r_3 \leq t$ in the specimen volume

$f_1(t)$– probability density function (PDF$_1$) of random chords with the length $2 \cdot r_1 \leq 2 \cdot t$ lying on the cutting plane

$f_2(t)$– probability distribution function (PDF$_2$) of random intersection radii with the length $r_2 \leq t$ lying on the cutting plane

$f_3(t)$– probability distribution function (PDF$_3$) of nodule radii with the length $r_3 \leq t$ in the specimen volume

$h_1$– distance from a nodule centre to a random chord
\( h_2 \) – distance from a nodule centre to a random cutting plane  
\( N_2 \) – the total number of cut nodules with the cross-section radius \( r_2 \geq t \)  
\( N_3 \) – volumetric grain density (grains per unit volume)  
\( r_1 \) – half-length of a random chord  
\( r_2 \) – radius of a random cross-section of a nodule  
\( r_3 \) – radius of a random nodule  
\( R_{\text{max}} \) – the largest radius of a nodule in the probe  
\( \overline{S} \) – the mean external surface area (estimated value) of nodular particles with different sizes  
\( t \) – length parameter  
\( x \) – integration variable

1. Introduction and model assumption

The problem of the mapping of an unknown probability distribution for the size of spherical particles in a non-transparent substance, is one of the classical stereological tasks. The first known solution of this task was proposed by Wicksell [1]. In this paper, a new approach towards this task, for a practical solution, will be presented. In order to derive mathematical formulas, the following assumptions have been used:

- In the specimen volume (Fig. 1) spherical particles with different diameters are redistributed having the volumetric grain density of \( N_3 \) (grains per unit volume).
- Spatial distribution of the grain centers is ruled by the Poisson statistical distribution.
- The probability of intersection between particles is negligible.
- The radii distribution is described by an unknown probability density function (PDF) or by a cumulative distribution function (CDF).
- The radius of the biggest particle in the probe does not exceed \( R_{\text{max}} \) value.

For a random section of the probe, the distance \( h_2 \) from the particle center to the cutting plane (see Fig. 2) has a uniform statistical distribution. In the same way, the distance \( h_1 \) between a random straight test-line (chord) and the centre of the particles has a uniform random variable.

![Fig. 1. Particles in the probe space near a cross-section](image)

Let us assume a few things: The number of particles with the radius \( r_3 \leq t \) is shown by the equation \( F_3(t) \) (the CDF of the grain size). The fraction of visible circular sections, lying on the random cutting plane, with a size of \( r_2 \leq t \), is described by this function \( F_2(t) \) (designated as CDF2). Whereas the fraction of chords placed inside the particles and having the length of \( 2 \cdot r_1 \leq 2 \cdot t \), is shown by the function \( F_1(t) \) designated further as CDF1.

The largest section radius as well as the largest half-length of a chord cannot be bigger than the value of \( R_{\text{max}} \) of the largest sphere.

Each above-mentioned function \( F_i(t) \) corresponds to a probability density function \( f_i(t) \) (denoted as PDF):

\[
 f_i(t) = \begin{cases} 
 0 & t < 0 \\
 \frac{dF_i}{dt} & 0 \leq t < R_{\text{max}} \\
 0 & t \geq R_{\text{max}} 
\end{cases} 
\]  

where \( i = 1 \) for the half-length of a chord, \( 2 \) for the radii of a cross-section, and \( 3 \) for the radii of a nodule.

![Fig. 2. Illustration of a nodule intersected by a random plane.](image)

Legend: \( r_1 \) – half-length of a random chord; \( r_2 \) – radius of a nodule intersection; \( r_3 \) – a nodule radius; \( h_1 \) – distance from a nodule center to a chord; \( h_2 \) – distance from a nodule center to a random plane

2. Planimetric analysis

Between the nodule radius \( r_3 \) and the radius of the section \( r_2 \), there is a relation \( r_3^2 = r_2^2 + h_2^2 \). Getting a section with the radius \( r_2 \geq t \) on a nodule with \( r_3 \) radius demands the fulfillment of two conditions:

\[
 r_3 \geq t \quad (2)
\]

\[
 h_2 \leq \sqrt{r_3^2 - t^2} \quad (3)
\]
For the uniform spatial distribution of the particles, the probability of meeting the relation (3) is equal to:

\[ P(h_2 \leq \sqrt{r_3^2 - t^2}) = \frac{\sqrt{r_3^2 - t^2}}{r_3} \]  

In the unit volume of the sample space, let us create a certain class of nodules with radii ranging from \( r_1 \) to \( r_3 + dr_3 \). Let us denote this class by \( C3 \). The cardinality \( d_n_{C3} \) of this set can be estimated by using the PDF3:

\[ d_n_{C3}(r_3) = N_3f_3(r_3)dr_3 \]  

If in this sample space the random cutting plane is placed, it cuts only those nodules, among the class \( C3 \), the distance of which from their centre and the cutting plane on both sides is less than \( r_t \). This new subset will be named as \( C2 \). The cardinality \( d_n_{C2} \) per unit area of the class \( C2 \) can be calculated as follows:

\[ d_n_{C2}(r_3) = 2r_3N_3f_3(r_3)dr_3 \]  

Among the class \( C2 \) the section radii \( r_2 \geq t \) will have only the particle which centers are placed at the distance less than \( \sqrt{r_3^2 - t^2} \) on the both side of the cutting plane. These particles will establish a new class \( C1 \) and its cardinality \( d_n_{C1} \) can be calculated by multiplication of the Eq. (6) by probability (4):

\[ d_n_{C1}(r_3, r_t) = 2\sqrt{r_3^2 - t^2}N_3f_3(r_3)dr_3 \]  

The total number \( N_2(t) \) of the sections with the radius \( r_2 \geq t \) can be obtained by integration of the above equation over nodules radius at the range from \( t \) to \( R_{max} \):

\[ N_2(t) = 2N_3 \int_t^{R_{max}} f_3(x)\sqrt{x^2 - t^2} \, dx \]  

Whereas the number \( N_2 \) per unit area of all nodules which are visible on this section can be determined by the integration of Eq. (6) at the range from \( 0 \) to \( R_{max} \) (or by integration of Eq. (8) from 0 to \( R_{max} \):

\[ N_2 = 2N_3 \int_0^{R_{max}} f_3(x) \, dx \]  

As it follows from the above equation, the share of the visible sections with the radius \( r_2 \geq t \) on the random plane should be equal to:

\[ 1 - F_2(t) = \frac{\int_t^{R_{max}} f_3(x)\sqrt{x^2 - t^2} \, dx}{\int_0^{R_{max}} f_3(x) \, dx} \]  

The integral in the denominator in the Eq. (10) for any statistical distribution function gives the expected (mean) value \( E[r_3] \) of the random variable \( r_3 \). For that reason, the CDF2 has the following form:

\[ F_2(t) = \frac{\int_t^{R_{max}} f_3(x)\sqrt{x^2 - t^2} \, dx}{E[r_3]} \]  

According to this relation (1) the equation for PDF2 can be obtained by differentiation of the Eq. (11): For the calculation of the integral in the numerator of the above formula, the Leibnitz integral rule was used. According to this rule, the derivative of the integral is equal to:

\[ \frac{d}{dt} \left( \int_t^{R_{max}} f_3(x)\sqrt{x^2 - t^2} \, dx \right) = -\int_t^{R_{max}} f_3(x) \frac{t}{\sqrt{x^2 - t^2}} \, dx \]  

With regard to (12) the form of PDF2 is:

\[ f_2(x) = \frac{t}{E[r_3]} \int_t^{R_{max}} f_3(x) \frac{t}{\sqrt{x^2 - t^2}} \, dx \]  

Having the form of \( f_2(x) \), which is estimated by quantitative metallography results, the equation (13) can be solved implicitly with respect to the \( f_3(x) \) function. In this task such a solution (named sometimes as inverse) according to [2] gives unsatisfactory results. That is why the numerical solutions of this task are used most often. An analysis of the planar section is based directly on Wickessell equation (13) which has been used for volume size distribution of spheroidal particles by Scheil [3], Schwartz [4], Saltykov [5, 6], Li at al. [7]. A similar solution for the mineralogy task has been presented in [8]. Unfortunately, small numerous errors of the empirically estimated function \( f_2(t) \) result in the „arbitrarily large perturbations of the solution“ [9, 10]. In this context, Eq. (13) is used usually not for designing the PDF3 form, but for the examination of matching the empirical function and one of the selected statistical distribution laws, e.g. normal, log-normal, Weibull or uniform-sized [2, 7, 11].
3. Linear analysis

The following relation \( r_s^2 = r_3^2 + h_1^2 \) links the nodular particle radius \( r_3 \) and half-length of the random chord \( r_1 \) on the random plane, where \( h_1 \) is the distance from the particle center to the chord line (see Fig. 2). The nodules of radius \( r_3 \) will have a chord length \( 2r_1 \geq 2r \) under the following conditions:

\[
\begin{align*}
  r_3 & \geq t \\
  h_1 & \leq \sqrt{r_3^2 - t^2}
\end{align*}
\]  
(14)
\( (15) \)

For uniform spatial distribution of particles, the probability of relation (15) is equal to:

\[
P(h_1 \leq \sqrt{r_3^2 - t^2}) = \frac{r_3^2 - t^2}{r_3^2}
\]  
(16)

Cardinality of C3 class particles is estimated by Eq. (5). The section number of C3 class particles by a random straight line per unit length of this line should be estimated as follows:

\[
d_{n_1}(r_3) = \pi r_3^2 N_3 f_3(r_3) d r_3
\]  
(17)

The number of chords longer than \( 2t \) for C3 particles can be calculated by the multiplication of the previous equation by probability (16):

\[
d_{n_1}(t, r_3) = \pi (r_3^2 - t^2) N_3 f_3(r_3) d r_3
\]  
(18)

The total amount \( N_1(t) \) of the bisectant longer than \( 2t \) should be obtained by the integration of the above equation over nodules radius at the range from \( t \) to \( R_{max} \):

\[
N_1(t) = \pi N_3 \int_t^{R_{max}} (x^2 - t^2) f_3(x) \, dx
\]  
(19)

The total amount of all visible bisectants at the random plane section should be determined by the integration of Eq. (19) with an inferior limit of integration equal to 0 (or by integration of Eq. (17) from 0 to \( R_{max} \)):

\[
N_1 = \pi N_3 \int_0^{R_{max}} x^2 f_3(x) \, dx
\]  
(20)

As it follows from the above formula, a fraction of the chords longer than \( 2t \), among all chords which are visible on the nodular particles on the random plane, should be equal to:

\[
1 - F_1(t) = \frac{\int_t^{R_{max}} (x^2 - t^2) f_3(x) \, dx}{\int_0^{R_{max}} x^2 f_3(x) \, dx}
\]  
(21)

The mean external surface (estimated value) of the nodular particles of different sizes is equal to:

\[
\bar{S} = E[S_3] = 4\pi \int_0^{R_{max}} f_3(x) x^2 \, dx
\]  
(22)

Thence, relation (21) should be transformed to form:

\[
\bar{S} - 4\pi \int_t^{R_{max}} f_3(x) (x^2 - t^2) \, dx = \frac{F_1(t)}{S}
\]  
(23)

According to relation (1) the equation for PDF \( f_1 \) should be obtained by differentiation of the Eq. (23). According to Leibnitz integral rule, the derivative of the integral in Eq. (23) is equal to:

\[
\frac{d}{dt} \left( \int_t^{R_{max}} f_3(x) (x^2 - t^2) \, dx \right) = -2t \int_t^{R_{max}} f_3(x) \, dx
\]  
(24)

Taking into account the derivative (24), the following form of the equation for PDF \( f_1 \) should be proposed as:

\[
f_1(t) = \frac{8\pi t}{S} \int_t^{R_{max}} f_3(x) \, dx
\]  
(25)

According to the PDF \( f_1 \) definition, the integral equation (25) may be transformed to form:

\[
f_1(t) = \frac{8\pi t}{S} \left[ F(R_{max}) - F_3(t) \right]
\]  
(26)

Because \( F_3(R_{max}) = 1 \):

\[
f_1(t) = \left[ 1 - F_3(t) \right] \frac{8\pi t}{S}
\]  
(27)

The above means that the empirical CDF \( f_1 \) should be estimated on the basis of the empirical estimation of PDF \( f_1 \) as follows:
\[ F_3(t) = 1 - \frac{\bar{S}}{8\pi t} f_1(t) \]  

(28)

The derivations of Eq. (28) with respect to \( t \) parameter gives:

\[ f_3(t) = \frac{\bar{S}}{8\pi} \left[ f_1(t) - \frac{d f_1(t)}{dt} \right] \]  

(29)

This formulation corresponds to known solutions of Cahn and Fullmann [12], Lord and Willis [13], and Spektor [14]:

\[ N_3 f_3(t) = \frac{N_1}{2\pi} \left[ f_1(t) - \frac{d f_1(t)}{dt} \right] \]  

(30)

4. Practical rules of the analysis

As it follows from Eq. (27), for \( F_3(t) < 1 \) (or \( t < R_{\text{max}} \)) at the intervals \( (t_1, t_2) \), when condition \( df_3(t)/dt = 0 \) is fulfilled, value of \( \text{PDF}_3 \) will be proportional to the \( t \) parameter. This condition is true when there are no particles of radius \( t_1 \leq t \leq t_2 \) in the space of the probe. In such a situation, the proportionality factor will depend on the particle’s share of radius less than \( t_1 \) and on the mean external surface of the nodules:

\[ k(t) = \frac{f_1(t)}{t} = \left[ 1 - F_3(t) \right] \frac{8\pi}{\bar{S}} \]  

(31)

For \( t_1 = 0 \) and a good resolution of image analysis (this means that optical resolution is much better than \( t_2 \)) it is possible to estimate the value of the mean external surface of nodules as follows:

\[ \bar{S} = \frac{8\pi t_2}{f_1(t_2)} \]  

(32)

As it follows from Eq. (27), regardless of the nature of the CDF, \( \lim_{t \to R} f_1(t) = 0 \).

Accuracy of \( \bar{S} \) estimation by means of Eq. (30) should be verifed by the integration of:

\[ \bar{S} = 4\pi \int_0^{R_{\text{max}}} t^2 f_3(t) \, dt \]  

(33)

5. Conclusions

- Empirical form of CDF_3 of the sizes of nodular particles should be estimated on the basis of the empirical measurement of PDF_1 on the statistical distribution of the length of random chords by Eq. (28).
- If in the tested probe there are no nodular particles of radii \( t_1 \leq t < t_2 < R_{\text{max}} \) then at the range between \( t_1 \) and \( t_2 \) the PDF_3 value is proportional to \( t \).
- If in the space of the analyzed probe there are no spheroids of radii less than \( t_2 \), then in the range between \( 0 \) and \( t_2 \) the PDF_1 value is proportional to \( t \). The proportionality factor is equal to \( 8\pi/\bar{S} \), where \( \bar{S} \) is the mean value of the external surface of the nodules.

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